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Bounded Fluctuations and Translation Symmetry Breaking: A Solvable Model

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The variance of the particle number (equivalently the total charge) in a domain of length $\mathscr{L}$ of a one-component plasma (OCP) on a cylinder of circumference W at the reciprocal temperature $\beta = 2$, is shown to remain bounded as $\mathscr{L} \to \infty$. This exactly solvable system with average density ρ has a density profile which is periodic with period $(\rho W)^{-1}$ along the axis of the infinitely long cylinder. This illustrates the connection between bounded variance and periodicity in (quasi) one-dimensional systems.[1] When $W \to \infty$ the system approaches the two-dimensional OCP and the variance in a domain Λ grows like its perimeter $|\partial\Lambda|$. In this limit, the system is translation invariant with rapid decay of correlations.

KEY WORDS: Coulomb systems; bounded fluctuations; translation symmetry breaking.

1. INTRODUCTION AND SUMMARY OF PREVIOUS WORK

Aizenman, Goldstein, and Lebowitz[1] have shown that bounded fluctuations in a one-dimensional one-component particle system imply the existence of a periodic structure. The present note illustrates this general property in a quasi one-dimensional system: the two-dimensional one-component plasma on the surface of a cylinder. This system is exactly solvable at the reciprocal temperature $\beta = 2$, where it was studied by Choquard, Forrester, and Smith.[2, 3]

The finite volume model is defined as follows. There are N particles of unit charge on the surface of a cylinder of circumference W and length L.

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The coordinates of a particle are $\mathbf{r}=(x, y)$ such that $-L/2 \leqslant x \leqslant L/2$, $-W/2 \leqslant y \leqslant W/2$. It is convenient to use also the complex coordinate $z=x+iy$.

The interaction energy $\phi(\mathbf{r}_1, \mathbf{r}_2)$ between two particles at $\mathbf{r}_1$ and $\mathbf{r}_2$ is a two-dimensional Coulomb potential required to be periodic in y with period W:

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = -\ln \left| 2 \sinh \frac{\pi(z_2 - z_1)}{W} \right| \tag{1}$$

(this is equivalent to Eq. (4) of ref. 3). There is also a neutralizing background and the full Hamiltonian also includes particle-background and background-background interactions. At small distances $|\mathbf{r}_2 - \mathbf{r}_1| \ll W$ the interaction (1) behaves like the two-dimensional Coulomb interaction $-\ln |\mathbf{r}_2 - \mathbf{r}_1|$, while at large distances along the cylinder $|x_2 - x_1| \gg W$ it behaves like the one-dimensional Coulomb interaction $-(\pi/W)\,|x_2 - x_1|$. At the special value, $\beta = 2$ of the inverse temperature, the model is exactly solvable. It can be regarded as a variant of the strictly one-dimensional one-component plasma,[4–6] with however a more explicit solution.

Choquard *et al.* calculated the one and two-particle distribution functions and their limits when N, $L \to \infty$, where $\rho = N/LW$ stays constant. In this limit the one-particle distribution function, i.e., the density, is a periodic function of x with period $1/\rho W$: it is a sum of equidistant identical Gaussians. The location of the centers of the Gaussians depends on the way the $L \to \infty$ limit is taken. If this limit is defined by the sequence of odd values of N, then one of the Gaussians stays centered at $x=0$. If, on the contrary, the thermodynamic limit is defined by the sequence of even values of N, the whole array of Gaussians is shifted by half a period and the origin is in the middle between two Gaussians. (It is also possible to take limits along sequences which don't keep the density exactly constant or use periodic boundary conditions along the x-axis. The latter would yield a translation invariant density corresponding to a uniform superposition of periodic ones).

In the following we shall, without loss of generality, concentrate on the case of odd values of N. Defining reduced values of the coordinates $\zeta = \rho W x$, $\lambda = (y_2 - y_1)/W$, and putting $\xi = \rho W^2$, the limit along such an odd sequence yields[3] the density

$$n(x) = \frac{(2\xi)^{1/2}}{W^2} \sum_{l=-\infty}^{\infty} \exp[-2\pi(\zeta - l)^2/\xi] \tag{2}$$

and the truncated two-particle distribution function

$$\rho^T(\mathbf{r}_1, \mathbf{r}_2) = -\frac{2\xi}{W^4} \exp[-\pi(\zeta_2-\zeta_1)^2/\xi] \times \left| \sum_{l=-\infty}^{\infty} \exp\left[-2\pi \left(\frac{\zeta_1+\zeta_2}{2} - l \right)^2 \Big/ \xi \right] + 2\pi i l \lambda \right|^2 \tag{3}$$

In the limit $W \to \infty$, the sums on l in (2) and (3) can be replaced by integrals and it can be easily checked that one recovers the expressions appropriate for a two-dimensional system in the whole plane, i.e., $n(x) = \rho$ and[7] $\rho^T = -\rho^2 \exp[-\pi\rho(\mathbf{r}_2 - \mathbf{r}_1)^2]$.

2. VARIANCE OF THE CHARGE IN AN INTERVAL

We now calculate the variance of the net charge, which here is equal to the variance $\langle N_I^2 \rangle - \langle N_I \rangle^2$ of the number of particles N_I in some interval I of the cylinder of circumference W and length $\mathscr{L}$. We show that this variance remains uniformly bounded as $\mathscr{L}$ increases and has a limit when $\mathscr{L} \to \infty$ along specified sequences.

After integration upon the y coordinates, one obtains the one-dimensional density $Wn(x)$ and the truncated two-particle density (when convenient, we use the reduced variable $\zeta = \rho W x$ instead of x):

$$\begin{aligned} R^T(\zeta_1, \zeta_2) &= \int_{-W/2}^{W/2} dy_1 \int_{-W/2}^{W/2} dy_2 \, \rho^T(\mathbf{r}_1, \mathbf{r}_2) \\ &= -\frac{2\xi}{W^2} \exp[-\pi(\zeta_2 - \zeta_1)^2/\xi] \sum_{l=-\infty}^{\infty} \exp\left[-4\pi \left(\frac{\zeta_1+\zeta_2}{2} - l \right)^2 \Big/ \xi \right] \end{aligned} \tag{4}$$

For simplicity, we choose the extremities of the interval I at the centers of two of the Gaussians, e.g., I is the interval $0 \leqslant \zeta \leqslant m$, where m is some positive integer. The variance in I is then given by

$$\begin{aligned} &\langle N_I^2 \rangle - \langle N_I \rangle^2 \\ &\quad = m + \int_0^{m/\rho W} dx_1 \int_0^{m/\rho W} dx_2 \, R^T(\zeta_1, \zeta_2) \\ &\quad = m - \frac{4}{\xi} \int_0^m du \exp[-\pi u^2/\xi] \int_{u/2}^{m-(u/2)} dv \sum_{l=-\infty}^{\infty} \exp[-4\pi(v-l)^2/\xi] \end{aligned} \tag{5}$$

where $u = \zeta_2 - \zeta_1$ and $v = (\zeta_1 + \zeta_2)/2$, and the symmetry between ζ_1 and ζ_2 has been taken into account.

The integral on v can be written as the difference between integrals in the ranges $(-(u/2), m-(u/2))$ and $(-(u/2), (u/2))$. The first one, when combined with the sum on l, like in (7), gives m times the integral of a Gaussian from $-\infty$ to ∞ which has a simple value. In the second one, one takes into account that the sum on l is an even function of v. One finally obtains for the variance in I,

$$\langle N_I^2 \rangle - \langle N_I \rangle^2 = m \left(1 - 2\xi^{-1/2} \int_0^m du \exp[-\pi u^2/\xi] \right) + \frac{8}{\xi} \int_0^m du \exp[-\pi u^2/\xi] \int_0^{u/2} dv \sum_{l=-\infty}^{\infty} \exp[-4\pi(v-l)^2/\xi] \tag{6}$$

The variance (6) is clearly bounded uniformly in m and its limit as $m \to \infty$ is given by

$$\langle N_I^2 \rangle - \langle N_I \rangle^2 \xrightarrow[m \to \infty]{} \frac{8}{\xi} \int_0^\infty du \exp[-\pi u^2/\xi] \int_0^{u/2} dv \sum_{l=-\infty}^{\infty} \exp[-4\pi(v-l)^2/\xi] \tag{7}$$

(the first term in the r.h.s. of (6) goes to 0 as $m \to \infty$).

3. SMALL AND LARGE CIRCUMFERENCE LIMITING CASES

The variance (7) depends on the cylinder circumference W through $\xi = \rho W^2$. Let us investigate limiting cases.

3.1. Small Circumference

For $W \to 0$, only the $l = 0$ term contributes to (7). The resulting double integral can be evaluated by rescaling the variables as $u\xi^{-1/2} \to u$ and $2v\xi^{-1/2} \to v$, and using the symmetry of the integrand in the new u, v coordinates, with the result

$$\lim_{W \to 0} [\langle N_I^2 \rangle - \langle N_I \rangle^2] = \tfrac{1}{2} \tag{8}$$

The result (8) has a simple interpretation. For small ξ, each Gaussian in (2) becomes a narrow peak while R^T is such that there is one particle in each peak and these particles are otherwise uncorrelated. The system behaves like a one-dimensional one-component plasma at zero temperature since the interaction behaves like $-(\pi/W)\,|x_2-x_1|$ with $W\to 0$. The variance is thus entirely due to the particles located near each end of the interval I. Each of these particles has a probability 1/2 of being inside I and thus contributes 1/4 to the variance of N_I.

3.2. Large Circumference

For large ξ, the sum on l in (7) can be replaced by an integral which is just $\xi^{1/2}/2$. One finds a variance which increases with W as

$$\langle N_I^2\rangle - \langle N_I\rangle^2 :\sim (\rho^{1/2}/\pi)\, W \tag{9}$$

This is the variance expected[8] for a large two-dimensional one-component plasma at $\beta=2$ with a boundary length $2W$.

4. ANOTHER CHOICE OF INTERVAL

The choice of intervals of Section 3 is likely to generate the largest possible variance. For illustrating that the variance in a large interval I is very sensitive to the precise positions of the extremities, we now choose these extremities in the middle between two adjacent Gaussians. For instance, I is the range $1/2\leqslant\zeta\leqslant m+(1/2)$. Calculations similar to the ones in Section 3 now lead to

$$\begin{aligned}\langle N_I^2\rangle - \langle N_I\rangle^2 &= m\left(1-2\xi^{-1/2}\int_0^m du\,\exp[-\pi u^2/\xi]\right)\\ &\quad+\frac{4}{\xi}\int_0^m du\,\exp[-\pi u^2/\xi]\int_{(-u+1)/2}^{(u+1)/2} dv \sum_{l=-\infty}^{\infty}\exp[-4\pi(v-l)^2/\xi]\end{aligned} \tag{10}$$

In the limit $m\to\infty$,

$$\langle N_I^2\rangle - \langle N_I\rangle^2 \to \frac{4}{\xi}\int_0^\infty du\,\exp[-\pi u^2/\xi]\int_{(-u+1)/2}^{(u+1)/2} dv \sum_{l=-\infty}^{\infty}\exp[-4\pi(v-l)^2/\xi] \tag{11}$$

For small ξ, (11) is dominated by the terms $l=0$ and $l=1$, and the variance has a factor $\exp(-\pi/\xi)$ which vanishes exponentially as $W\to 0$.

This is as expected, since the density vanishes as a Gaussian tail at the extremities of the interval I, through which particles enter or leave I.

For large W, the variance is still given by (9). In that limit the density oscillations disappear and the precise locations of the extremities of I become irrelevant.

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